On the non-linear mechanics of hydrodynamic stability

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SUMMARY

In most work on the theory of stability of laminar flow, infinitesimal disturbances only have been considered, so that only the *initial* growth of the disturbance has been determined. It is the object of the present paper to extend the theory to larger amplitudes and to study the mechanics of disturbance growth with the inherent non-linearity of the hydrodynamical system taken into account.

The Reynolds stress (where averages are taken with respect to some suitable space coordinate) is the fundamental consequence of the non-linearity, and its effects can be anticipated as follows. Initially a disturbance grows exponentially with time according to the linear theory, but eventually it reaches such a size that the transport of momentum by the finite fluctuations is appreciable and the associated mean stress (the Reynolds stress) then has an appreciable effect on the mean flow. This distortion of the mean flow modifies the rate of transfer of energy from the mean flow to the disturbance and, since this energy transfer is the cause of the growth of the disturbance, there is a modification of the rate of growth of the latter.

It is suggested that, in many cases, an equilibrium state may be possible in which the rate of transfer of energy from the (distorted) mean flow to the disturbance balances precisely the rate of viscous dissipation of the energy of disturbance. A theory based on certain assumptions about the energy flow is given to describe both the growth of the disturbance and the final equilibrium state, and application is made to the cases of Poiseuille flow between parallel planes and flow between rotating cylinders. The distorted mean flow in the equilibrium state can be calculated and from this, in the latter case, the torque required to maintain the cylinders in motion. Good agreement is obtained with G. I. Taylor's measurements of the torque for the case when the inner cylinder rotates and the outer cylinder is at rest.

1. INTRODUCTION

The immediate objective of the theory of hydrodynamic stability is to understand the mechanism of instability in laminar flow and to obtain a criterion for its occurrence. A more fundamental objective is to understand how, and under what circumstances, turbulence may arise from laminar instability. The connection between laminar instability and turbulence may not be a direct one, but under certain circumstances instability of laminar flow will be a necessary prelude to transition to turbulence. It is clear that the stability problem in its general form must be considered to be non-linear, because the equations of motion are non-linear.

The mathematical problem of hydrodynamic stability can be formulated by taking the given steady-state solution of the equations of motion and superimposing a disturbance of a suitable kind; this results in a set of (non-linear) 'disturbance' equations, which govern the behaviour of the disturbance. If the solution of the equations shows that any disturbance ultimately decays to zero, the flow is said to be stable; whereas if the disturbance can be permanently different from zero, the flow is unstable. It does not always happen that instability leads to turbulent motion, because another (possibly more complex) form of laminar motion may be the result. Indeed, it will be shown that this is often the case.

Naturally, the solution of the disturbance differential equations is simplified considerably by linearization for small disturbances, and for a description of theories based on this assumption the reader is referred to the book by Lin (1955). On the basis of linear theory it is possible to consider disturbances which contain an exponential time factor of the form $\exp(kt)$, t being the time. The boundary conditions on the disturbance equations require the *vanishing* at the boundaries of quantities like the disturbance velocity components. Consequently, the boundary conditions are homogeneous, and there is an eigenvalue problem for the determination of the quantity k. In this (linear) case, stability or instability is defined as follows: if it is possible for k to have a positive real part, the flow is unstable; otherwise the flow is stable.

The prediction by linear theory of a disturbance which increases exponentially with time is a feature which has occasionally given rise to the suggestion that turbulence would necessarily ensue from the growth of the disturbance to large amplitudes; however, examples are known in which this is not the case. On the other hand, it has been argued that the non-linear terms will stabilize completely a flow which is unstable according to linear theory, but such arguments can generally be refuted (Stuart 1956a). In this paper, certain features of the role played by the non-linear terms of the equations of motion are discussed, and this leads to a clarification of the connection between linear and non-linear instability theories. Attention is restricted to flows which have constant local Reynolds number.

In cases of instability of fluid flow, the disturbance is usually periodic in at least one direction of space. Thus it is convenient to take averages with respect to one of the spatial dimensions, and to separate the flow into a mean part and a disturbance part, where the latter has zero mean*. It is clear that the two parts of the flow are interdependent through the action of the Reynolds stress (arising from the disturbance) on the mean flow. (For a discussion of Reynolds stresses, the reader is referred to the books by Goldstein (1938) and Townsend (1956).) On the basis of linear theory, the disturbance is assumed to be so small that the effect of the Reynolds stress on the mean motion can be neglected, in which case the mean flow is the original laminar flow. However, in a non-linear theory the interdependence of the mean and disturbance parts of the flow must be taken into account. Let us now consider a flow whose local Reynolds number does not vary, as in the case of flow between parallel planes or concentric cylinders, and let the flow be perturbed by a small disturbance. Initially the disturbance amplifies exponentially with time according to linear theory, but eventually it reaches such a size that the mean transport of momentum by the finite fluctuations is appreciable, and then the associated mean stress (the Reynolds stress) has an appreciable effect on the mean flow. This distortion of the mean flow clearly modifies the rate of transfer of energy from the mean flow to the disturbance and, since this energy transfer is the cause of the growth of the disturbance, there is a modification of the rate of growth of the latter. These processes, in which the disturbance distorts the mean flow and the distortion of the mean flow modifies the rate of growth of the disturbance, occur simultaneously.

It is natural to enquire if an equilibrium state is possible, in which the rate of transfer of energy from the mean flow to the disturbance balances precisely the rate of viscous dissipation of energy of the disturbance. In such an equilibrium state, the disturbance will have a definite finite amplitude and the mean flow will be distorted from its original laminar form. Experimental evidence of an equilibrium state of this kind is afforded by G. I. Taylor's observations on the instability of flow between rotating cylinders, where the instability takes the form of cellular, toroidal vortices spaced regularly along the axes of the cylinders. Taylor (1923, p. 342) observed that "A moderate increase in the speed of the apparatus merely increased the vigour of the circulation in the vortices without altering appreciably their spacing or position", and suggested that "The experiments ... indicate that the effect of the second-order [non-linear] terms is to prevent

* It should be mentioned that, an another formulation of the problem, the 'disturbance' is defined to be the whole of the deviation from the original laminar flow. However, with this definition the 'disturbance' must contain a 'mean' part, a fact which was overlooked by several authors in their studies of non-linear instability theory (Stuart 1956 a). In the opinion of the writer, this approach does not yield such a clear understanding of the physical processes involved in instability as does the approach based on the concept of a mean flow and a disturbance flow (with zero mean) interacting through the action of a Reynolds stress. Consequently, in this paper the flow will always be separated into a mean part and a disturbance part (with zero mean). The term 'disturbance' will *not* be used to denote the deviation from the original laminar flow.

the vortices from increasing indefinitely in activity". The combination of the mean flow with a disturbance of definite amplitude may be referred to as an *equilibrium* flow. Examples relating to Poiseuille flow between parallel planes and to flow between rotating cylinders are described in § 2 and § 3, and for the case of flow between cylinders good agreement with experiment is obtained. For the case of Poiseuille flow, however, there is no experimental evidence that an equilibrium flow of the kind described above does occur. There is the possibility in this case, as in other cases, that a flow of this kind does not occur, but rather that there is a continual generation of harmonics of the basic disturbance and of other disturbances.

In the discussion above, attention has been paid to equilibrium flows which may develop when the original laminar flow is unstable according to linear theory. Some flows, however, such as Couette flow between parallel planes and Poiseuille flow in a circular pipe, are completely stable against infinitesimal disturbances. Even so, turbulence can occur at sufficiently high Reynolds numbers. Furthermore, turbulence occurs in some flows (for example, in Poiseuille flow between parallel planes) at a lower Reynolds number than the critical according to linear theory, that is, it occurs when the laminar flow is stable with respect to infinitesimal disturbances. A suggestion which may lead to an explanation of such phenomena is that the appropriate laminar flow may be unstable with respect to finite disturbances. When a disturbance of suitable magnitude is present, the mean flow may be distorted to such a form that the rate of transfer of energy to the disturbance can balance exactly its rate of dissipation by viscosity. On the other hand, a finite disturbance which is small enough will presumably decay to zero amplitude, either because the rate of energy transfer to the disturbance is insufficient to balance the rate of viscous dissipation of kinetic energy, or because the energy transfer is actually from the disturbance to the mean flow. As an example of instability for finite disturbances, Meksyn & Stuart (1951) considered the case of Poiseuille flow between parallel planes and showed that the critical Reynolds number drops as the amplitude of the disturbance rises. The reader is referred also to Lin (1955) and Stuart (1956 b) for a discussion of these ideas. A discussion of the formulation of the instability problem for plane Couette flow with finite disturbances was given by Noether (1921).

It will be convenient to refer to non-linear disturbances as existing under *supercritical* conditions if the Reynolds number is above the value which is critical for linearized instability, and as existing under *subcritical* conditions if the Reynolds number is such that the flow is stable with respect to infinitesimal disturbances. A non-linear disturbance may clearly arise spontaneously under supercritical conditions as a result of continued amplification of a secondary disturbance; the sequence of events which would lead to a non-linear disturbance under subcritical conditions is less evident and will not be discussed here.

If equilibrium flow is established consisting of a mean flow with a steady-amplitude finite disturbance, it does not follow that it is a stable flow at all Reynolds numbers above the critical value for which it first occurs. It is likely, however, to be stable for a certain range of Reynolds number above the critical. Moreover, it is clear that it must become unstable at some Reynolds number if turbulence is to occur at higher Reynolds numbers. For example, in the case of flow between rotating cylinders mentioned above, Taylor observed that "a large increase [in the angular speed] caused the symmetrical motion [of the cellular vortices] to break down into some kind of turbulent motion, which it was impossible to follow by eye". Another interesting case is that of Poiseuille flow between parallel planes, where the critical Reynolds number according to linear theory is a Reynolds number for which the flow is normally turbulent. Consequently, equilibrium flows under supercritical conditions are, in this case, almost certainly unstable. Similar considerations apply to equilibrium flows under subcritical conditions, at least for the higher Reynolds numbers for which they are valid.

The development of a non-linear instability theory for boundary-layer flows, where the flow and local Reynolds number change in the stream direction, presents additional difficulties. Whereas in the linear instability theory it is permissible to regard a boundary-layer flow as nearly parallel and to neglect boundary-layer growth, it does not seem obvious that such an approximation is permissible in a non-linear theory. Because the local Reynolds number increases in the downstream direction, any disturbance is convected into regions of higher Reynolds number and the effect of this continuous change of Reynolds number would have to be taken into account. Consequently, the non-linear theory which will be described in this paper for flows with constant local Reynolds number does not necessarily apply quantitatively to the case of the boundary layer. The main features of the theory are of wide applicability, but there may be additional factors influencing them because of the growth of the thickness of the boundary layer.

An interesting suggestion concerning the development of turbulence from the growth of small disturbances has been made by Landau (1944). As noticed above, the occurrence of instability in a flow may lead to the replacement of the original laminar flow by a new laminar flow, which consists of a mean flow with a superimposed finite disturbance. This flow may be expected to persist for a certain range of Reynolds number above the critical value and then to become unstable at some Reynolds number against a new (second) type of disturbance. A new equilibrium flow, consisting of a mean flow with two superimposed modes of disturbance, is then conceivable for a range of Reynolds number above the second critical value. As the Reynolds number is raised still further, additional modes of disturbance may appear successively until, at sufficiently large Reynolds number, the flow is so highly disturbed as to be considered turbulent. In the case of flow between rotating cylinders, experiments show that the development of turbulence takes place fairly slowly as the Reynolds number is raised; this would correspond to the first two at least of the successive critical Reynolds numbers of Landau's theory being fairly widely spaced. On the other hand, there are cases of flow in which turbulence develops rather suddenly as the Reynolds number is raised, and in these cases one might infer that the critical Reynolds numbers are close together.

In the discussion given above, attention has been entirely directed to instability in fluids which are in motion. There is, however, another important type of instability, namely that which occurs when a horizontal layer of fluid is heated from below, causing an unstable density gradient in the fluid. Instability takes the form of convection in cells of polygonal planform, and occurs when a certain parameter (the Rayleigh number) is above a critical value. The linear theory of instability is well understood (see, for example, Pellew & Southwell 1940) and is known to be analogous, under certain conditions, to the theory of instability of flow between rotating cylinders. The fundamental physical process in the instability is the conversion of potential energy associated with the gravitational field into kinetic energy of the convective disturbance motion. When the disturbance has a finite amplitude, the mean heat transport by the convective motion causes a modification of the mean temperature distribution, where averages are taken over horizontal planes; therefore, because of the corresponding modification of the mean density distribution, the rate of transfer of energy from the gravitational field to the disturbance is modified. It appears that a steady (equilibrium) state is possible, in which the rate of transfer of energy into the disturbance balances precisely the rate at which energy is dissipated and diffused. The non-linear theory of thermal convection has been studied by Sorokin (1954) and by Malkus & Veronis (1958), both by methods related to those of the present paper and by perturbation series expansions.

Some of the results obtained in this paper are similar to results obtained from the turbulence models due to Burgers (1948) and Hopf (1956). For a discussion of non-linear instability effects in the Burgers model, the reader is referred to a paper by Stuart (1956b).

2. Disturbances under supercritical conditions in Poiseuille flow between parallel planes

In this section a simple treatment of the non-linear problem for disturbances under supercritical conditions is described, and it is applied in this section and in § 3 to the cases of Poiseuille flow between parallel planes and of flow between rotating cylinders. The method is based essentially on a balance of energy, taking into account the mean-flow distortion due to the Reynolds stress, and it is the 'integral' properties of the flow which are treated rather than the spatial details. Consequently, it is clear that the method has an obvious application to cases of instability, such as that of flow between rotating cylinders, where it is the overall properties of the flow (as opposed to the details) which are important. On the other hand, in cases of instability which are governed by the

Orr–Sommerfeld equation (for example, the case of plane Poiseuille flow), the instability depends very much upon precise details of the mean flow, especially in the vicinity of the critical layer. Consequently, the method is less valid for such cases, and the application to plane Poiseuille flow has less validity than the case of flow between rotating cylinders. Moreover, there is less experimental evidence of quantitative value in the former case than in the latter. In spite of these facts, the application to plane Poiseuille flow is of considerable illustrative value, and is therefore presented in this section before a treatment in § 3 of the case of flow between rotating cylinders, for which a comparison with experiment will be made.

Consider a two-dimensional flow between parallel planes, and let x_1 denote the coordinate parallel to the planes and x_2 the coordinate normal to them, with u_1, u_2 denoting the corresponding components of velocity, Let P denote pressure, ρ density, ν kinematic viscosity and t the time. Now let us introduce a reference length L, equal to half the distance between the planes, and a reference speed U_0, the maximum speed in laminar flow. Thus we define

$$x_1 = Lx, \quad x_2 = Lz, \quad u_1 = U_0 u, \quad u_2 = U_0 w, \quad P = p\rho U_0^2, \quad t = Lt/U_0. \tag{2.1}$$

Then the Navier–Stokes equations and equation of continuity are

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} &= -\frac{\partial p}{\partial x} + \frac{1}{R}\nabla^2 u, \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z} + \frac{1}{R}\nabla^2 w, \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0, \end{aligned}\right\} \tag{2.2}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2$, and $R = U_0 L/\nu$ denotes the Reynolds number for the problem.

The solution for undisturbed laminar Poiseuille motion is

$$\bar{u} = 1 - z^2, \qquad \bar{w} = 0, \qquad \bar{p} = -2x/R + \text{const.}, \tag{2.3}$$

where a bar above a quantity denotes a mean value with respect to the distance x. This is the basic laminar motion whose non-linear instability characteristics we wish to examine.

Now let

$$u = \bar{u}(z, t) + u'(x, z, t), \qquad w = w'(x, z, t), \tag{2.4}$$

where u' and w' denote components of a finite disturbance (with zero mean). Since the disturbance is finite, $\bar{u}$ is no longer given by (2.3) but is distorted by the Reynolds stress. Furthermore, if the disturbance is growing or decaying in amplitude, $\bar{u}$ depends on time. We suppose that u' and w' are harmonic in x with wavelength $2\pi/\alpha$ and are given by Fourier series. If we substitute (2.4) into (2.2), separate out the mean and disturbance parts of the motion and then integrate the resulting disturbance equations to obtain the energy balance relation, we have (without approximation)

$$\frac{\partial}{\partial \tau}\iint \tfrac{1}{2}(u'^2+w'^2)\,dxdz = \iint (-\overline{u'w'})\frac{\partial \bar{u}}{\partial z}\,dxdz - \frac{1}{R}\iint \left(\frac{\partial w'}{\partial x}-\frac{\partial u'}{\partial z}\right)^2 dxdz,$$
$$= I_1 - I_2/R, \tag{2.5}$$

where the integrands are evaluated over a volume bounded by the planes and by one wavelength. The mean velocity $\bar{u}$ occurring in (2.5) is given by

$$\frac{\partial \bar{u}}{\partial \tau} + \frac{\partial \overline{u'w'}}{\partial z} = -\frac{\partial \bar{p}}{\partial x} + \frac{1}{R}\frac{\partial^2 \bar{u}}{\partial z^2}. \tag{2.6}$$

For a discussion of the derivation of the above equations, and for the non-linear disturbance differential equations for u' and w', the reader is referred to Stuart (1956 a, b). Equations (2.5) and (2.6) will be used here as an approximate basis for the solution of the non-linear problem of the growth of the disturbance (u', w').

We note that in equation (2.5) the term on the left-hand side gives the rate of growth of the disturbance energy within the volume considered. On the right-hand side, the term I_1 is the integral of the product of the Reynolds stress and the mean velocity gradient, and represents the rate of energy transfer from the mean flow to the disturbance; the term I_2/R is always positive and represents the rate of viscous dissipation of energy of the disturbance. If I_1 is positive and greater than I_2/R, the disturbance energy is growing and the disturbance is increasing in amplitude. Equation (2.6) shows how the distribution of mean motion is affected by the viscous stress, pressure gradient, and Reynolds stress due to the disturbance. An equilibrium flow is possible if $\bar{u}$ can be so distorted by the Reynolds stress that $I_1 = I_2/R$ for a given Reynolds number.

It is worth considering at this point the boundary conditions on the mean motion. For the velocity they are $\bar{u} = 0$ at $z = \pm 1$. We shall also adopt the condition that the mean pressure gradient shall remain unchanged; this is clearly physically realistic, since the pressure gradient is externally applied*. As a consequence, the mean skin friction on the walls will be the same in the equilibrium flow as in the laminar flow. However, the mean velocity near the channel centre must drop when a disturbance is present, because energy is required to maintain the disturbance. The unchanged pressure gradient is given by equation (2.3) and the mean motion equation thus takes the form

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial}{\partial z}(\overline{u'w'}) = \frac{2}{R} + \frac{1}{R}\frac{\partial^2 \bar{u}}{\partial z^2}. \tag{2.7}$$

In a state of equilibrium, $\bar{u}$ is independent of time and equation (2.7) is easily integrated to yield

$$\bar{u}(z) = 1 - z^2 + R\int_1^z \overline{u'w'}\,dz. \tag{2.8}$$

* Other boundary conditions, such as that of constancy of mass flux, yield qualitatively similar results for the amplitude of oscillation.

Since $\overline{u'w'}$ is an odd function of z, $\bar{u} = 0$ at $z = \pm 1$. Moreover, since I_1 is positive, $\overline{u'w'}$ must be of opposite sign to $\partial \bar{u}/\partial z$ in at least a dominant range of z; therefore, it must be dominantly positive when z is positive. Thus $\bar{u}$ is less than unity at $z = 0$ and, in general, $\bar{u}(z)$ is everywhere less than its value for laminar flow. A corollary of the condition on the mean pressure gradient is that the Reynolds number is based on a velocity U_0 which is the maximum for laminar flow with the same mean pressure gradient.

Suppose we now define a stream function for the disturbed flow in the form

$$\psi = \phi_0(z, t) + \phi_1(z, \tau) e^{i\alpha(x - c_r \tau)} + \tilde{\phi}_1(z, \tau) e^{-i\alpha(x - c_r \tau)} + \phi_2(z, \tau) e^{2i\alpha(x - c_r \tau)} + \\ + \tilde{\phi}_2(z, \tau) e^{-2i\alpha(x - c_r \tau)} + \ldots, \qquad (2.9)$$

where the symbol $\sim$ represents a complex conjugate. ψ represents a mean flow together with a periodic disturbance consisting of the first harmonic with wavelength $2\pi/\alpha$, and higher harmonic components having wave-numbers $n\alpha$ (n integral) but the same (real) wave velocity c_r, which is assumed to be independent of time. The amplification or damping of the disturbance, and the consequent changes in the mean velocity (given by $\bar{u} = \partial \phi_0 / \partial z$), are accounted for by the dependence of all the ϕ-functions on τ.

If we substitute (2.9) into (2.2), utilize

$$u = \partial \psi / \partial z, \qquad w = -\partial \psi / \partial x, \qquad (2.10)$$

eliminate the pressure and separate out the Fourier components of the disturbance, we obtain a set of equations for the functions $\bar{u}$, ϕ_1, ϕ_2, and so on. The equations are similar to (3.2), (3.3) and (3.4) of the paper by Stuart (1956b), except that allowance must be made for the dependence of the functions on τ. Because of the non-linearity of the system, all of the functions are mutually dependent and an infinite set of equations has to be solved to obtain an exact solution. In particular, the mean-motion equation is (2.6) with

$$\overline{u'w'} = i\alpha\{\phi_1' \tilde{\phi}_1 - \tilde{\phi}_1' \phi_1 + 2(\phi_2' \tilde{\phi}_2 - \tilde{\phi}_2' \phi_2) + \ldots\}, \qquad (2.11)$$

where primes denote differentials with respect to z.

In the linear theory of instability, the Reynolds stress is neglected, so that the mean velocity is the laminar Poiseuille flow. Furthermore, the stream function (2.9) contains only the functions ϕ_0 and ϕ_1, the real part of which determines the actual stream function. The problem then is the solution of the Orr–Sommerfeld equation (see Lin 1955), namely

$$(\bar{u} - c_r - ic_i)(\phi_1'' - \alpha^2 \phi_1) - \bar{u}'' \phi_1 = -(i/\alpha R)(\phi_1^{\text{iv}} - 2\alpha^2 \phi_1'' + \alpha^4 \phi_1), \quad (2.12)$$

where $\bar{u} = 1 - z^2$ and the function ϕ_1 is proportional to $\exp(\alpha c_i \tau)$. The non-linear problem posed here is to follow the growth of a disturbance at a given Reynolds number and, in particular, to find out if an equilibrium flow is possible at that Reynolds number. To this end, the stream function (2.9) has been chosen to allow for changes in the basic disturbance, partly through the generation of harmonics, due to the non-linearity.

A simple approximate method of solution

The non-linear effects of a disturbance of finite amplitude appear to be of two kinds. On the one hand there is, through the Reynolds stress, a modification of the energy transfer between the mean flow and the first harmonic component, ϕ_1, of the disturbance. On the other hand, higher harmonics (ϕ_2, etc.) of the disturbance are generated and there is an interchange of energy between them and both the mean flow and the first harmonic component. The assumption which will be made here is that the dominent non-linear interaction is that between the mean flow and the first harmonic component of the disturbance. To put it in another way, the effect of the Reynolds stress on the mean flow is taken into account, thus causing modifications of the energy transfer between the mean flow and the first-harmonic disturbance, but the generation of higher harmonics is ignored. A similar assumption was made by Meksyn & Stuart (1951) for disturbances under subcritical conditions. In a formulation of this kind there are two differential equations to determine $\bar{u}$ and ϕ_1, namely, (2.7) and (2.12) with c_i replaced by $\alpha^{-1}\partial/\partial\tau$; in equation (2.11), $\phi_2 = \phi_3 = \ldots = 0$.

In the present case of disturbances under supercritical conditions, we shall make the additional assumption that the disturbance (u', w') is similar in *shape* to a solution given by linear theory, but that the solution is multiplied by an amplitude factor, $a(\tau)$, which is a function of time. Then the approximate problem is to satisfy the disturbance energy-balance equation (2.5) and the equation of mean motion (2.7), in which case the amplitude is determined. If the amplitude of the disturbance in an equilibrium flow is required, $\bar{u}$ is given in terms of $\overline{u'w'}$ by (2.8) and can be substituted into (2.5). With the 'shape' assumption, (2.5) then gives a relation between a^2 and a^4, so that a^2 is determined in terms of integrals involving functions given by linear theory. This is the most important result.

An equation illustrating the *growth* of the disturbance may be obtained by noting that the term $\partial\bar{u}/\partial\tau$ in (2.7) is negligible at small times on the basis of linear theory, and also at large times because it is to be expected that a steady mean flow will be approached. If, on this basis, the term $\partial\bar{u}/\partial\tau$ is ignored at all times, (2.5) and (2.8) yield the equation

$$\gamma_1\frac{\partial a^2}{\partial\tau} = \gamma_2\alpha a^2 - \gamma_3\alpha^2 R a^4 - \frac{\gamma_4 a^2}{R}, \tag{2.13}$$

where a is the amplitude of the disturbance and

$$\left.\begin{aligned} \gamma_1 &= \int_0^1 \{|\phi'|^2+\alpha^2|\phi|^2\}\,dz, & \gamma_2 &= 4\int_0^1 z(\phi_i\phi_r'-\phi_r\phi_i')\,dz,\\ \gamma_3 &= 4\int_0^1 (\phi_i\phi_r'-\phi_r\phi_i')^2\,dz, & \gamma_4 &= 2\int_0^1 \{|\phi''|^2+2\alpha^2|\phi'|^2+\alpha^4|\phi|^2\}\,dz, \end{aligned}\right\} \tag{2.14}$$

the suffixes r and i denoting real and imaginary parts. The function $\phi(z)$ is the amplitude distribution of a disturbance stream function according to linear theory. Equation (2.13) is of the form

$$\frac{dy}{d\tau} = \beta_1 y - \beta_2 y - \beta_3 y^2, \tag{2.15}$$

where y is the square of the amplitude of the disturbance. Equation (2.15) was given earlier by Landau (1944), although he does not state how, and under what assumptions, it was derived.

The terms in (2.15) and the corresponding ones of (2.13) have the following physical meanings. The term $dy/d\tau$ represents the rate of change of the kinetic energy of the disturbance, the term $\beta_1 y$ the rate of energy transfer from the mean flow to the disturbance, the term $\beta_2 y$ the rate of viscous dissipation of the disturbance, and the non-linear term $\beta_3 y^2$ the restriction of the rate of energy transfer to the disturbance by the Reynolds-stress distortion of the mean flow. For very small disturbances, the non-linear term is negligible, and if $\beta_1 > \beta_2$, the disturbance amplifies like $C\exp[(\beta_1-\beta_2)\tau]$, where C is a constant. This solution corresponds to linear stability. On the other hand, if the Reynolds stress is included the general solution of equation (2.15) is

$$y = \frac{Ce^{(\beta_1-\beta_2)\tau}}{1+\left(\dfrac{\beta_3}{\beta_1-\beta_2}\right)Ce^{(\beta_1-\beta_2)\tau}}. \tag{2.16}$$

When $\tau \to -\infty$, $y \to C\exp[(\beta_1-\beta_2)\tau]$, which is the linearized solution given above. Furthermore, when $\tau\to+\infty$, $y\to(\beta_1-\beta_2)/\alpha\beta_3$, whatever the value of C. This solution suggests that, whatever the initial size of the disturbance, it always builds up to the same limiting amplitude. This conclusion is physically reasonable, although it should be remembered that the exact form of (2.16) is based on the assumption that $\partial\bar{u}/\partial\tau$ may be neglected at all times.

Reverting now to (2.13), we note that the critical Reynolds number is

$$R_c = \gamma_4/\alpha\gamma_2,$$

the amplification factor of ϕ^2 on linear theory is

$$2\alpha c_i = \beta_1-\beta_2 = \frac{\gamma_2\alpha-\gamma_4/R}{\gamma_1},$$

and the square of the equilibrium amplitude is

$$a_e^2 = \frac{\beta_1-\beta_2}{\beta_3} = \frac{\gamma_2\alpha-\gamma_4/R}{\gamma_3\alpha^2 R} = \frac{\gamma_2(R-R_c)}{\gamma_3\alpha R^2}.$$

Thus a_e is proportional to the square root of the difference between the actual Reynolds number and the critical Reynolds number. In order to calculate specific values of the amplitude at a given Reynolds number, it is necessary to approximate to the function ϕ. A fairly obvious choice would be the function appropriate to the minimum critical Reynolds number. No calculation of this function has been made, although Thomas (1953) has calculated and tabulated ϕ for $\alpha = 1, R = 10^4, c = c_r+ic_i = 0{\cdot}2375+0{\cdot}0037i$. The function is illustrated in figure 1, $\mathscr{R}$ denoting the real part and $\mathscr{I}$ the imaginary part. Numerical integration of (2.14) then gave

$$\gamma_1 = 2{\cdot}05146,\quad \gamma_2 = 0{\cdot}040192,\quad \gamma_3 = 0{\cdot}002308,\quad \gamma_4 = 247{\cdot}62. \tag{2.17}$$

Using these values, we find that $\alpha R_c = 6150$, which is in reasonable agreement with Thomas's critical value, namely, $R_c = 5780$ at $\alpha = 1{\cdot}02$. Thus, even though the function ϕ is not the correct stream function at the critical Reynolds number, the energy-balance relation yields a fairly good approximation to the critical Reynolds number. It is suggested, therefore, that the numbers (2.17) may be applicable over a wide range of R, and possibly also for small variations of α. Another characteristic of the linear instability theory which can be calculated from (2·17) is the amplification constant c_i at $\alpha = 1$, $R = 10^4$, and this is found to be $c_i = 0{\cdot}00376$. The closeness of this to Thomas's value, 0·0037, serves as a check on the accuracy of the numerical work.

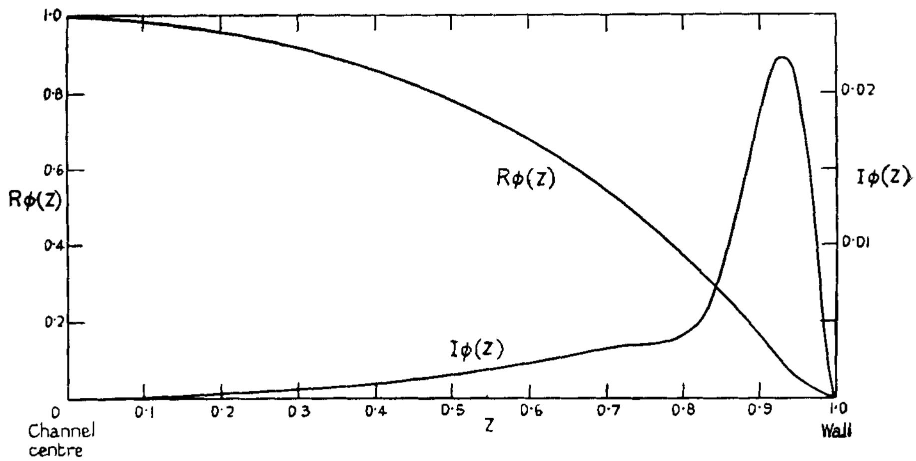

Figure 1. Function ϕ for Poiseuille flow at $R = 10^4$, $\alpha = 1$, $c = 0{\cdot}2375 + 0{\cdot}0037i$.

For the non-linear properties, we consider a wave-number, $\alpha = 1$, since this is close to the value at the critical Reynolds number. The equilibrium amplitude number, a_e, rises from zero at the critical Reynolds number (6150) through 0·00256 at $R = 10^4$ to a maximum value of 0·00266 at $R = 12300$ ($= 2R_c$). To obtain the actual (dimensionless) stream function at a given value of R, the curves in figure 1 should be multiplied by the appropriate value of a. The distribution of Reynolds stress is shown in figure 2 (obtained from Thomas's calculations), from which the actual (dimensionless) Reynolds stress can be obtained by multiplying by a^2.

Using the calculations described above, the mean velocity profile of the equilibrium flow can be calculated from equation (2.8); for $R = 10^4$, it has the form shown in figure 3. The distorted profile is everywhere less than the laminar profile because of the energy transfer to the disturbance, but has the same gradient at the wall. It can readily be shown that the work done by the pressure gradient to maintain the motion precisely balances the total viscous dissipation of energy in the equilibrium flow. The root-mean-squares of the disturbance velocity components, $(\overline{u'^2})^{1/2}$ and $(\overline{w'^2})^{1/2}$, are

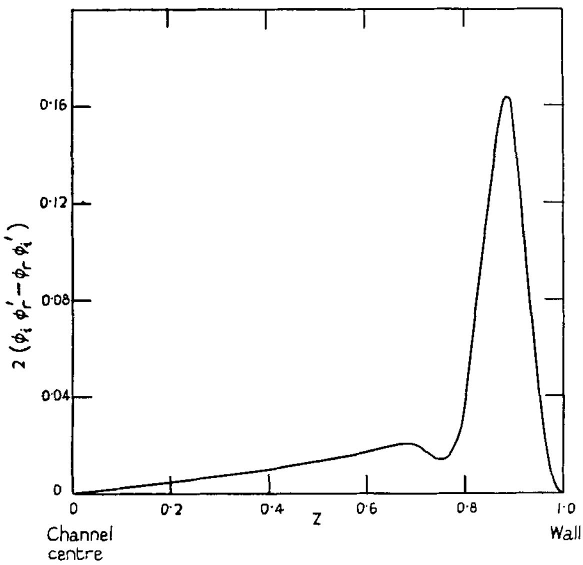

Figure 2. Reynolds stress function for Poiseuille flow at $R = 10^4$.

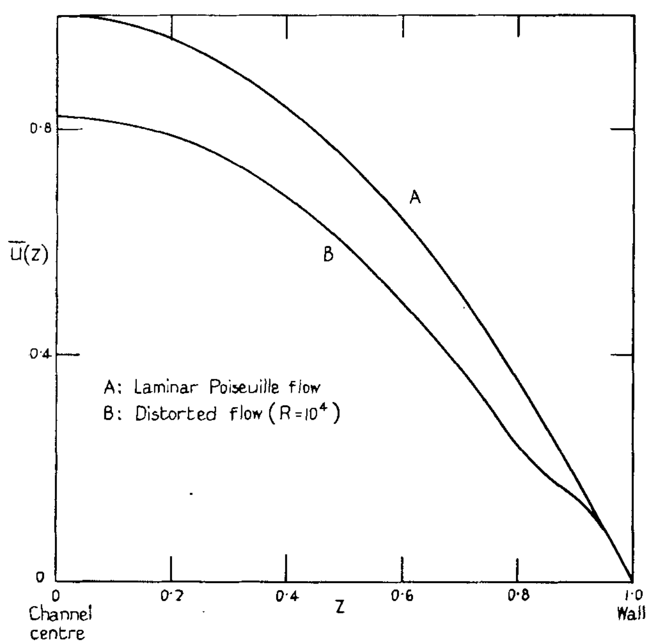

Figure 3. Comparison of laminar and distorted mean flows.

shown in figure 4 for $\alpha = 1$, $R = 10^4$. The magnitudes are similar to those applicable in turbulent motion at the same Reynolds number. A further point worthy of note is that, at a Reynolds number of 10^4, the maximum value of the Reynolds stress is of the same order as the maximum value of the viscous stress.

As mentioned in the Introduction, the equilibrium flow under supercritical conditions in Poiseuille flow between parallel planes is almost certainly unstable, because turbulence normally exists at those Reynolds numbers for which the supercritical flow exists. However, it is of interest that the magnitude (see figure 4) of disturbance which can be sustained is similar to the magnitude appropriate to turbulent flow at the same Reynolds number (Reichardt 1938; Laufer 1950).

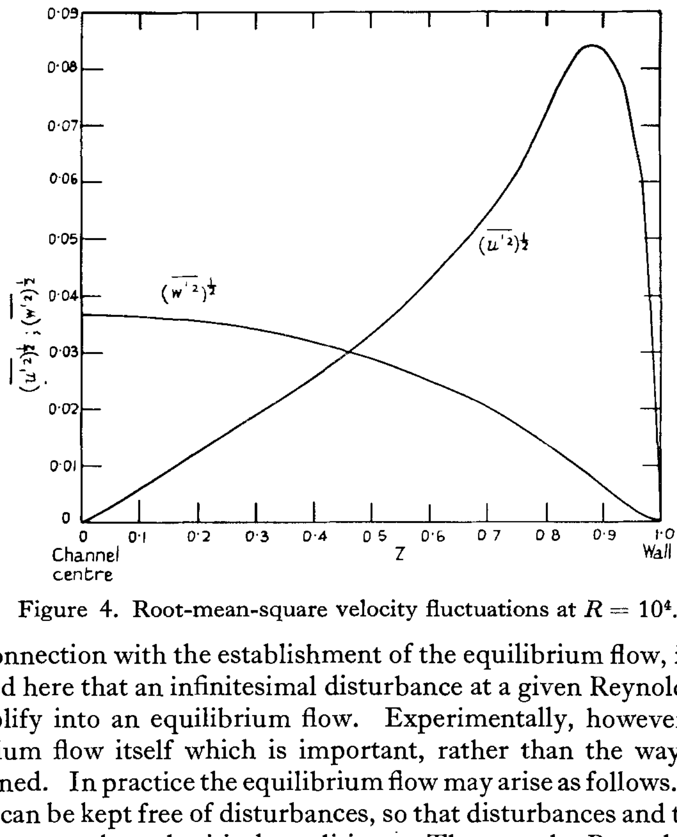

Figure 4. Root-mean-square velocity fluctuations at $R = 10^4$.

In connection with the establishment of the equilibrium flow, it has been suggested here that an infinitesimal disturbance at a given Reynolds number can amplify into an equilibrium flow. Experimentally, however, it is the equilibrium flow itself which is important, rather than the way in which it is attained. In practice the equilibrium flow may arise as follows. Suppose the flow can be kept free of disturbances, so that disturbances and turbulence do not occur under subcritical conditions. Then, as the Reynolds number is raised, a weak disturbance will appear at the critical Reynolds number. The disturbance will have a small but finite amplitude. Then, as the Reynolds number is increased, the flow could (if it were not unstable) retain the same wavelength as the disturbance which first occurred, but with an amplitude $a \sim R^{-1}(R-R_c)^{1/2}$. This idea is discussed again in § 3 in connection with the flow between rotating cylinders.

3. Disturbances under supercritical conditions in flow between rotating cylinders

The second application of the theory will be to the flow between two concentric rotating cylinders, with r, θ, z as the cylindrical coordinates and u, v, w the corresponding velocity components. It is assumed that the flow has rotational symmetry and is therefore independent of θ. Then the Navier–Stokes and continuity equations are

$$\left.\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\nabla^2-\frac{1}{r^2}\right)u,\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r} &= \nu\left(\nabla^2-\frac{1}{r^2}\right)v,\\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w,\\
\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z} &= 0,
\end{aligned}\right\} \tag{3.1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}. \tag{3.2}$$

It is known from the work of Taylor (1923) and others that, when the rotational speed of the inner cylinder (or, to be more specific, a parameter sometimes called the Taylor number) is above a critical value, the steady laminar flow is unstable. The disturbance which appears takes the form of cellular, toroidal vortices spaced regularly along the axis of the cylinders. The linear instability theory for this flow is well known (Taylor 1923) and our purpose here is to study the non-linear theory, primarily in order to obtain the amplitude of the equilibrium flow.

Since the flow which results from the instability is periodic with respect to z, it is convenient to take averages with respect to z. In order to allow for the distortion of the disturbance by the non-linearity, we write

$$u = u' = u_1(r,t)e^{i\alpha z}+u_2(r,t)e^{2i\alpha z}+\ldots+\tilde{u}_1(r,t)e^{-i\alpha z}+\tilde{u}_2(r,t)e^{-2i\alpha z}+\ldots \tag{3.3}$$

together with exactly similar series for $v' = v-\bar{v}(r,t)$ and $w' = w$, where the symbol $\sim$ denotes a complex conjugate and a bar above a quantity denotes a mean value. If the above series are substituted into the equations of motion (3.1) and the Fourier components are separated, there results a set of equations for all the velocity functions involved in (3.3).

For the mean motion, we find

$$\frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'^2})-\frac{1}{r}(\overline{v'^2}+\bar{v}^2) = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{3.4}$$

$$\frac{\partial \bar{v}}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\overline{u'v'}) = \nu\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\bar{v}\,; \tag{3.5}$$

similar equations have been derived and discussed by Townsend (1956) in connection with turbulent flow. Equation (3.4) gives the radial pressure gradient required to balance the centrifugal force and Reynolds stress, while (3.5) gives the mean rotational velocity $\bar{v}$ as a function of the Reynolds stress. The dependence of $\bar{v}$ on t is retained because, if the disturbance is growing or decaying, the mean motion undergoes distortion in order to maintain the energy balance. A suitable boundary condition to apply to the mean flow is that just enough external power is supplied to maintain the angular velocities of the cylinders at constant values, even though the skin friction is changing. If r_1 and r_2 are the radii of the inner and outer cylinders and Ω_1 and Ω_2 their angular velocities, the boundary conditions are

$$\bar{v} = r_1 \Omega_1 \quad \text{at } r = r_1; \qquad \bar{v} = r_2 \Omega_2 \quad \text{at } r = r_2. \tag{3.6}$$

In a state of equilibrium, $\partial\bar{v}/\partial t$ is zero and (3.5) can be integrated to give

$$\bar{v} = Ar + \frac{B}{r} + \frac{r}{\nu}\int_{r_1}^{r} \frac{\overline{u'v'}}{r}\,dr, \tag{3.7}$$

where

$$A = \frac{\Omega_1(1-mr_2^2/r_1^2)}{1-r_2^2/r_1^2} + \frac{1}{\nu(r_1^2/r_2^2-1)}\int_{r_1}^{r_2} \frac{\overline{u'v'}}{r}\,dr, \tag{3.8}$$

$$B = \frac{r_1^2\Omega_1(1-m)}{1-r_1^2/r_2^2} + \frac{r_1^2}{\nu(1-r_1^2/r_2^2)}\int_{r_1}^{r_2} \frac{\overline{u'v'}}{r}\,dr, \tag{3.9}$$

and $m = \Omega_2/\Omega_1$. With zero Reynolds stress, these relations become those appropriate to laminar Couette flow between rotating cylinders.

In addition to the mean motion equations, it is possible to write down an infinite set of differential equations for the harmonic components of the disturbance. These components are all mutually dependent because of the non-linearity of the system. These equations will not be derived or considered in detail here, because it is proposed to use an approximation similar to that used in § 2. The assumption is that the dominant non-linear interaction is between the mean flow ($\bar{v}$) and the first harmonic component of the disturbance (u_1, v_1, w_1), where the latter is assumed to be similar in shape to that given by linear theory but multiplied by an amplitude factor. The mean flow is distorted by the Reynolds stress of the disturbance and the consequent alteration of the energy transfer between mean flow and disturbance determines the amplitude. As in § 2, the disturbance energy-balance relationship is required for this purpose.

If the equations of the mean motion are subtracted from the equations (3.1), equations are obtained for the disturbance velocities u', v', w' (as defined by (3.3)). From these it is possible to obtain the following disturbance energy equation, which is an exact consequence of the equations for an axisymmetric disturbance:

$$\frac{\partial}{\partial t}\iint \tfrac{1}{2}\rho(u'^2+v'^2+w'^2)r\,dr\,dz = \iint (-\rho\overline{u'v'})\left(\frac{\partial\bar{v}}{\partial r} - \frac{\bar{v}}{r}\right)r\,dr\,dz - \\ -\mu\iint (\xi'^2+\eta'^2+\zeta'^2)r\,dr\,dz, \tag{3.10}$$

the vorticity components of the disturbance being given by

$$\xi' = -\frac{\partial v'}{\partial z}, \quad \eta' = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial r}, \quad \zeta' = \frac{1}{r}\frac{\partial}{\partial r}(rv'). \tag{3.11}$$

The three integral terms of (3.10) can be interpreted in a similar way to those of (2.5). The net rate of increase of disturbance energy is equal to the difference between the integral of the product of the Reynolds stress and the flow shear $(\partial \bar{v}/\partial r - \bar{v}/r)$, which represents the rate of transfer of kinetic energy from the mean flow to the disturbance, and the rate of viscous dissipation of kinetic energy.

The approximate non-linear problem of stability is that of solving the equation of mean motion, (3.5), and the equation of energy balance for the disturbance, (3.10), where, in the latter, the disturbance is assumed to be specified in shape but not in amplitude. The pair of equations (3.5) and (3.10) can then be used to determine the amplitude, as in § 2. If the amplitude of the disturbance in the equilibrium flow is required, $\bar{v}$ is given in terms of $\overline{u'v'}$ by (3.7); then (3.10) involves both second and fourth powers of the amplitude, and the latter can therefore be determined. Moreover, an assumption similar to that of § 2 concerning the *growth* of the disturbance can be made. This assumption is that $\partial\bar{v}/\partial t$ can be ignored at all times on the grounds that it is certainly negligible at both small and large times.

By the procedure outlined above a differential equation of the form (2.15) is obtained, and it has the solution (2.16). In order to calculate the coefficients in the equation, it is necessary to specify the shape of the flow field so that several integrals can be evaluated. In the detailed calculations which follow, attention is restricted to the case $\Omega_2 = 0$ so that comparison can be made with experiments of Taylor (1936). Secondly, only the very simplest case is considered, that in which the annular gap is small compared with the radii of the cylinders. The linear stability problem (Chandrasekhar 1953) is then specified by

$$\left.\begin{aligned} &(D^2-\lambda^2)(D^2-\lambda^2-\sigma)^2 v_1 + \lambda^2 T v_1 = 0,\\ &v_1 = D^2 v_1 = D(D^2-\lambda^2-\sigma)v_1 = 0 \quad \text{at } \zeta = \pm\tfrac{1}{2}, \end{aligned}\right\} \tag{3.12}$$

where

$$\left.\begin{aligned} &d = r_2 - r_1, \qquad 2r_0 = r_1 + r_2, \qquad r = r_0 + \zeta d,\\ &\lambda = \alpha d, \qquad \sigma = kd^2/\nu, \qquad D \equiv d/d\zeta,\\ &T = \Omega_1^2 r_1 d^3/\nu^2, \qquad R = \Omega_1 r_1 d/\nu. \end{aligned}\right\} \tag{3.13}$$

The disturbance is proportional to $\exp(kt)$. The symbol T denotes the Taylor number and R denotes the Reynolds number.

To the same approximation ($d \ll r_0$) the mean flow is given by

$$\bar{v} = \tfrac{1}{2} r_1 \Omega_1 (1-2\zeta) - (a^2 r_1^3 \Omega_1^3 d^2/\nu^2) Q(\zeta), \tag{3.14}$$

$$Q(\zeta) = -2\int_{-1/2}^{\zeta} Z\, d\zeta + 2\zeta \int_{-1/2}^{1/2} Z\, d\zeta + \int_{-1/2}^{1/2} Z\, d\zeta, \tag{3.15}$$

$$Z = -SP, \quad P = (D^2-\lambda^2)S, \quad u_1 = -ar_1\Omega_1 P, \quad v_1 = (ar_1^2\Omega_1^2 d/\nu)S, \tag{3.16}$$

a being the amplitude of the velocity u_1. The flow shear is given to the same approximation by

$$\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r} = -\frac{r_1 \Omega_1}{d} - \frac{2a^2 r_1 \Omega_1^3 d}{\nu^2}\left(\int_{-1/2}^{1/2} Z\, d\zeta - Z(\zeta)\right). \tag{3.17}$$

It has been assumed that the Reynolds-stress term in (3.14) is much larger than the terms of order d/r_0.

If the approximation $d \ll r_0$ is adopted and a few transformations are applied, the largest terms of (3.10) yield

$$\gamma_1 \frac{da^2}{d\tau} = \gamma_2 a^2 - \gamma_3 R^2 a^4 - \frac{\gamma_4 a^2}{\lambda^2 T}, \tag{3.18}$$

where

$$\tau = 2\nu t/d^2, \tag{3.19}$$

$$\left.\begin{aligned} \gamma_1 &= \int_{-1/2}^{1/2} S^2\, d\zeta, \qquad \gamma_2 = \int_{-1/2}^{1/2} Z\, d\zeta, \\ \gamma_3 &= -2\left(\int_{-1/2}^{1/2} Z\, d\zeta\right)^2 + 2\int_{-1/2}^{1/2} Z^2\, d\zeta, \quad \gamma_4 = \int_{-1/2}^{1/2} [(D^2-\lambda^2)P]^2\, d\zeta. \end{aligned}\right\} \tag{3.20}$$

This equation is of the form (2.15) and has the property that the amplitude tends to a limiting value with the passage of time. If the Reynolds-stress term $(-\gamma_3 R^2 a^4)$ is neglected and the disturbance is neutral, equation (3.18) is simply Chandrasekhar's (1953) variational condition. The critical Taylor number is given by $\lambda^2 T_c = \gamma_4/\gamma_2$ and the equilibrium amplitude by

$$a_e^2 = \frac{\gamma_2}{\gamma_3 R^2}\left(1 - \frac{T_c}{T}\right). \tag{3.21}$$

Thus a_e is proportional to the square-root of the difference between the actual Taylor number and the critical Taylor number.

Equation (3.12) can be used to specify the function $S(\zeta)$ and, since we are primarily concerned with the equilibrium flow, $S(\zeta)$ will be assumed to be the function appropriate to $\sigma = 0$, $\lambda = \lambda_c = \pi$, $T = T_c = 1708$. This is the case of a disturbance at the critical Taylor number, for which we have, from the variational condition given by Chandrasekhar (1953),

$$S = -\frac{1}{8\pi^2} + \frac{1}{\pi^4} + \frac{\zeta^2}{2\pi^2} + \frac{\cos\pi\zeta}{4\pi^3} + \frac{A}{2\pi^3}(1 + \tfrac{1}{5}\cos 2\pi\zeta) - \frac{1}{\pi^4}\left(1 + \frac{2\pi}{5}\right) \times \frac{\cosh\pi\zeta}{\cosh\frac{1}{2}\pi} \tag{3.22}$$

to a good approximation, where $A = -0{\cdot}02686$. Then for a range of Taylor numbers above the critical value, and for the particular wave-number $\lambda = \pi$, the amplitude can be evaluated according to (3.20), (3.21) and (3.22). Thus

$$a_e^2 = \frac{5{\cdot}425 \times 10^4}{R^2}\left(1 - \frac{1708}{T}\right). \tag{3.23}$$

This relation, together with (3.16) and (3.22), then gives the velocity distribution of the disturbance.

Once the amplitude of the velocity distribution is known, the mean skin friction can be calculated from (3.17) and the mean torque on a cylinder of given length then follows. Taylor (1936) carried out experiments with cylinders 84·4 cm long, the outer cylinder having radius $r_2 = 4{\cdot}05$ cm while the radius of the inner cylinder varied from case to case. Here we shall compare the measured and calculated torque for the case $r_1 = 3{\cdot}94$ cm. The number of vortices spaced along the axis is 700 approximately and, since alternate vortices rotate in opposite directions, the torque measured is the mean torque, where the term 'mean' is defined in the sense of this paper. In figure 5, G denotes the torque measured in units gm cm^2 sec^{-2} and N the angular speed measured in units rev. sec^{-1}. It can be seen that the agreement is good for Taylor numbers (T being proportional

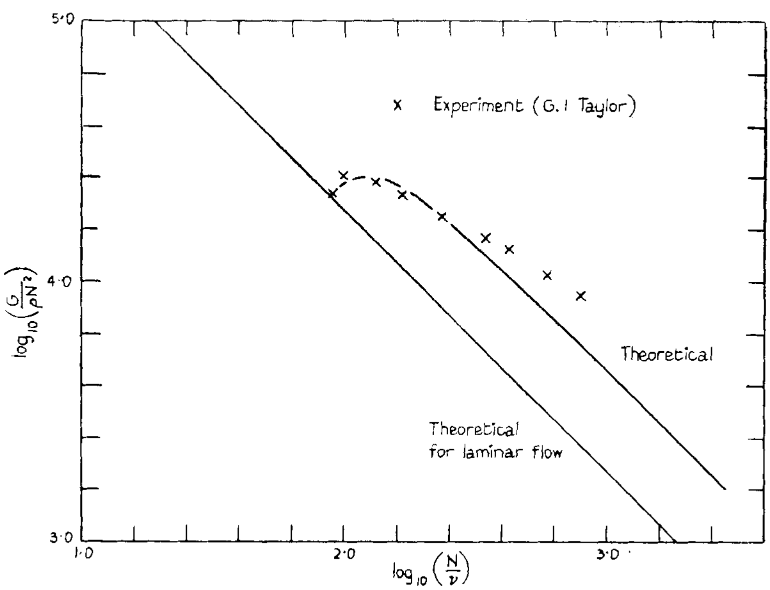

Figure 5. Comparison of theoretical and experimental torque for flow between rotating cylinders. Inner radius 3·94 cm, outer radius 4·05 cm.

to N^2) up to about ten times the critical value ($\log_{10} N/\nu = 1{\cdot}955$), since even at $\log_{10} N/\nu = 2{\cdot}5$ the divergence between theory and experiment is only about 7% of the torque. The gradual divergence is probably due to upper harmonics of the basic mode of disturbance. The mean velocity, calculated from (3.14) is shown in figure 6 for a Taylor number of about $2{\cdot}1 \times 10^4$. The reversal of sign of the velocity gradient in the centre may not be realistic, since this Taylor number is near the limit of validity of the theory. The increased velocity gradient at the cylinders is a consequence of the fact that the vortex motion requires more power than does laminar Couette flow at the same rotational speed.

From the nature of assumptions made in the present theory, particularly the 'shape' assumption, it may be expected that the method used above will be valid for some range of Taylor number above the critical value, and figure 5 shows that this is indeed the case. For a non-linear instability problem which has a certain similarity to the present one, that of thermal convection in a horizontal layer of fluid, Malkus & Veronis (1958) have devised a rigorous expansion procedure which gives the limiting steady amplitude of convection for a certain range of Rayleigh number above the critical value. Their method gives results very similar to those obtained by an integral procedure analogous to the present one. Presumably such an expansion would be valid for the cylinder problem also.

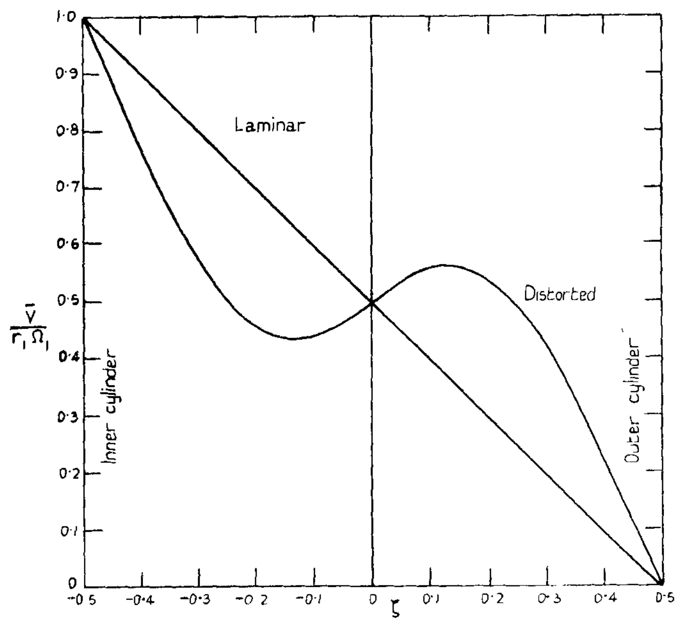

Figure 6. Mean flow between rotating cylinders ; $\log_{10}(N/\eta) = 2{\cdot}5$, $T = 20\,987$.

With regard to the way in which an equilibrium state may be set up at a Taylor number above the critical value, it has been suggested here (though not proved rigorously) that an infinitesimal disturbance which increases exponentially with time will necessarily tend to an equilibrium state. On the other hand, the experiments generally require a different interpretation concerning growth. Taylor (1923, p. 331) states that "The speed of the motor [driving the cylinders] was ... gradually increased till instability occurred" and (p. 342) that "A moderate increase in the speed of the apparatus merely increased the vigour of the circulation in the vortices...". It seems, therefore, that, as the Taylor number is raised, instability sets in at the critical Taylor number with a definite wavelength. Further increase of the Taylor number entails the flow retaining the same wavelength, but with the amplitude of the disturbance motion given by (3.23). Consequently, it seems that the non-linear stability problem of growth at a given Reynolds or Taylor number, as treated herein, does not correspond directly to the circumstances of Taylor's experiments whereas the treatment of the equilibrium flow presumably does.

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